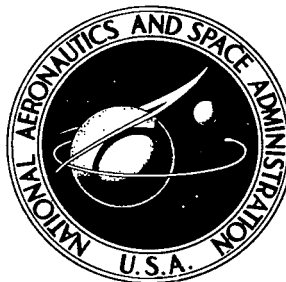
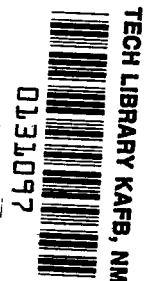


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# DERIVATION OF APPROXIMATE EQUATIONS FOR SOLVING THE PLANAR RENDEZVOUS PROBLEM

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*Langley Station, Hampton, Va.*



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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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# DERIVATION OF APPROXIMATE EQUATIONS FOR SOLVING THE PLANAR RENDEZVOUS PROBLEM

By Gene W. Sparrow and Douglas B. Price  
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## SUMMARY

The desirability of obtaining simplified and accurate relative equations of motion for two-impulse rendezvous application resulted in the examination of the Clohessy and Wiltshire type of simplified equation. An analysis of these equations revealed that large errors were possible if the radial separation of the two vehicles was large. In an effort to develop more accurate equations for rendezvous, the apparent erroneous terms were replaced by exact terms. The resulting modified equations contained terms not present in the Clohessy and Wiltshire equations. When the modified equations were applied to the problem of rendezvous, they were found to be generally superior to the Clohessy and Wiltshire equations, particularly for long transfer angles. The scope of the investigation is limited to the case in which the target vehicle is in a circular orbit coplanar with the ferry vehicle's orbit.

## INTRODUCTION

Of particular interest in the space program are the problems of rendezvous between two vehicles in orbit. Since the exact equations describing the relative motion of the vehicles are nonlinear, the usual approach in finding a solution is to approximate the exact equations. Clohessy and Wiltshire first obtained a linear approximation of these equations for the case in which the target vehicle is in a circular orbit (refs. 1, 2, and 3). Some forms of the linear equations that are even simpler, though less accurate, were obtained by modifying the gravity term (ref. 3). London (ref. 4) improved the accuracy of the linear form by including second-order terms in his equations. Anthony and Sasaki (ref. 5) and others generalized further by considering a slightly elliptical target orbit.

To obtain the impulses required for a two-impulse rendezvous, it is necessary to know the initial required velocity and the velocity at rendezvous. The linear equations of Clohessy and Wiltshire are sufficiently simple so that the two impulses may be defined explicitly in closed form. Although it is possible to define explicitly the two impulses for rendezvous in a second-order system (see ref. 5), the expressions are usually not

simple. Since the linear form is desirable for simplicity but is not accurate, an effort was made to find more accurate equations of this form for two-impulse rendezvous application. This paper presents the results of the investigation.

The scope of the investigation is limited to the case in which the target vehicle is in a circular orbit coplanar with the ferry vehicle's orbit. The first impulse is assumed to occur at the initiation of the problem, and the target vehicle's angular travel during the coast following the first impulse is arbitrarily limited to  $360^\circ$ . Examples are presented which illustrate the application of the equations derived herein to the problem of rendezvous.

## SYMBOLS

$d$	slant-range separation of ferry and target vehicles, kilometers
$d_R$	slant-range separation of ferry and target vehicles at intercept, kilometers
$k$	an arbitrary integer
$Q(\tilde{y}_0)$	function defined by equation (13)
$r_F$	ferry vehicle's distance from mass center of central body, kilometers
$r_T$	target vehicle's distance from mass center of central body, kilometers
$t$	elapsed time from problem initiation, seconds
$X, Y$	shell-coordinate reference-axis system (see fig. 1)
$x, y$	coordinates in the $X, Y$ axis system, kilometers
$X_i, Y_i$	inertial reference-axis system fixed in central body (see fig. 1)
$\left. \begin{array}{l} (\bar{x}_c)_{\text{error}} \\ (\bar{y}_c)_{\text{error}} \end{array} \right\}$	error expressions defined by equations (12)
$\Delta V_1$	magnitude of first velocity impulse, kilometers/second
$\Delta V_2$	magnitude of second velocity impulse, kilometers/second

$\delta$	symbol used to denote a trigonometric expression defined by equation (17)
$\mu$	product of universal gravitational constant and mass of central body
$\sigma_F$	angular position of ferry vehicle as measured in $X_i, Y_i$ axis system, radians
$\tau$	elapsed time from first impulse to intercept, seconds
$\phi$	angle defining the direction of the ferry vehicle with respect to the target vehicle, degrees (see fig. 1)
$\omega$	angular rate of target vehicle, radians/second
$\omega t$	angular position of target vehicle as measured in the $X_i, Y_i$ axis system, radians or degrees (see fig. 1)
$\omega \tau$	angular travel of target vehicle about central body during the interval extending from first impulse to intercept, radians or degrees

#### Subscripts:

c	circular component
o	initial conditions

An asterisk denotes a required initial condition.

Dots above symbols denote differentiation with respect to time or with respect to  $\omega t$  when nondimensional form is used.

A bar above a symbol denotes the nondimensional form.

## ANALYSIS

### Coordinate System and Equations of Motion

The equations of relative motion between the ferry and target vehicles are to be expressed in so-called shell coordinates. (See fig. 1.) The shell coordinate system was first introduced in reference 6. In this system the  $X$  and  $Y$  axes are centered in the target vehicle, which is moving in a circular orbit at a radius  $r_T$  with angular rate  $\omega$ . The ferry vehicle's position relative to the target vehicle is defined by the  $x$ -coordinate,

which is measured along a curved line of constant radius  $r_T$ , and by the  $y$ -coordinate, which is directed along  $r_F$  and measured from the target vehicle's radius  $r_T$ . The rendezvous maneuver is initiated at first impulse ( $\omega t = 0$ ) with the target vehicle located on the  $X_i$  axis and is completed when  $x = y = 0$  for some arbitrary value of  $\omega t$ . The symbol  $d$  denotes the slant-range separation of the ferry and target vehicles at any time.

The inertial positions of the ferry and target vehicles are defined by the polar coordinates  $r_F, \sigma_F$  and  $r_T, \omega t$ , respectively. The relationships between these polar coordinates and the shell coordinates are

$$\left. \begin{aligned} x &= r_T(\omega t - \sigma_F) & y &= r_F - r_T \\ \dot{x} &= r_T(\omega - \dot{\sigma}_F) & \dot{y} &= \dot{r}_F \\ \ddot{x} &= -r_T\ddot{\sigma}_F & \ddot{y} &= \ddot{r}_F \end{aligned} \right\} \quad (1)$$

The differential equations of motion of the ferry vehicle about a central body in polar coordinates are expressed by the familiar form

$$\left. \begin{aligned} \ddot{r}_F - r_F\dot{\sigma}_F^2 &= -\frac{\mu}{r_F^2} \\ r_F\ddot{\sigma}_F + 2\dot{r}_F\dot{\sigma}_F &= 0 \end{aligned} \right\} \quad (2)$$

The differential equations of motion in shell coordinates may be determined by substituting into equations (2) the appropriate terms found from equations (1). The exact differential equations of motion in shell coordinates thus obtained are

$$\left. \begin{aligned} \ddot{x} &= 2\left(1 + \frac{y}{r_T}\right)^{-1}\left(\omega - \frac{\dot{x}}{r_T}\right)\dot{y} \\ \ddot{y} &= r_T\left(1 + \frac{y}{r_T}\right)\left(\omega - \frac{\dot{x}}{r_T}\right)^2 - r_T\omega^2\left(1 + \frac{y}{r_T}\right)^{-2} \end{aligned} \right\} \quad (3)$$

The gravitational constant  $\mu$  has been eliminated from equations (3) by the substitution  $\mu = r_T^3\omega^2$ , since  $r_T\omega = \sqrt{\mu/r_T}$  for a circular target orbit at  $r_T$ .

It is convenient to use equations (3) in a nondimensional form which is obtained by nondimensionalizing displacements with respect to  $r_T$ , velocities with respect to the target velocity  $\omega r_T$ , and accelerations with respect to  $\omega^2 r_T$ . The nondimensional form of equations (3) is

$$\left. \begin{aligned} \ddot{\bar{x}} &= 2(1 + \bar{y})^{-1} \left(1 - \frac{\dot{\bar{x}}}{\dot{\bar{y}}}\right) \dot{\bar{y}} \\ \ddot{\bar{y}} &= (1 + \bar{y}) \left(1 - \frac{\dot{\bar{x}}}{\dot{\bar{y}}}\right)^2 - (1 + \bar{y})^{-2} \end{aligned} \right\} \quad (4)$$

### Approximate Differential Equations of Motion and Their Solution

When the terms  $(1 + \bar{y})^{-1}$ ,  $(1 + \bar{y})^{-2}$ , and  $\left(1 - \frac{\dot{\bar{x}}}{\dot{\bar{y}}}\right)^2$  in equations (4) are expanded in a binomial series and grouped according to their order, the following equations are obtained:

$$\left. \begin{aligned} \ddot{\bar{x}} &= 2\dot{\bar{y}} \left[ 1 - \left( \frac{\dot{\bar{x}}}{\dot{\bar{y}}} + \bar{y} \right) + \left( \bar{y} \frac{\dot{\bar{x}}}{\dot{\bar{y}}} + \bar{y}^2 \right) - \left( \bar{y}^2 \frac{\dot{\bar{x}}}{\dot{\bar{y}}} + \bar{y}^3 \right) + \dots \right] \\ \ddot{\bar{y}} &= \left( -2\frac{\dot{\bar{x}}}{\dot{\bar{y}}} + 3\bar{y} \right) + \left( \frac{\dot{\bar{x}}^2}{\dot{\bar{y}}^2} - 2\bar{y} \frac{\dot{\bar{x}}}{\dot{\bar{y}}} - 3\bar{y}^2 \right) + \left( \bar{y} \frac{\dot{\bar{x}}^2}{\dot{\bar{y}}^2} + 4\bar{y}^3 \right) + \dots \end{aligned} \right\} \quad (5)$$

If it is assumed that the ferry vehicle remains in the vicinity of the target vehicle during the rendezvous maneuver, the higher order terms of equations (5) will be small in comparison with the lower order terms and may be neglected. A first-order approximation of equations (5) is

$$\left. \begin{aligned} \ddot{\bar{x}} &= 2\dot{\bar{y}} \\ \ddot{\bar{y}} &= -2\frac{\dot{\bar{x}}}{\dot{\bar{y}}} + 3\bar{y} \end{aligned} \right\} \quad (6)$$

Equations (6) are the so-called nondimensionalized Clohessy and Wiltshire equations in shell coordinates. Equations (6) may be easily integrated with respect to time to obtain

$$\left. \begin{aligned} \bar{x} &= (4 \sin \omega t - 3\omega t) \dot{\bar{x}}_0 + 2(1 - \cos \omega t) \dot{\bar{y}}_0 + 6(\omega t - \sin \omega t) \bar{y}_0 + \bar{x}_0 \\ \bar{y} &= -2(1 - \cos \omega t) \dot{\bar{x}}_0 + (\sin \omega t) \dot{\bar{y}}_0 + (4 - 3 \cos \omega t) \bar{y}_0 \end{aligned} \right\} \quad (7)$$

where the initial conditions  $\bar{x}_0$ ,  $\bar{y}_0$ ,  $\dot{\bar{x}}_0$ , and  $\dot{\bar{y}}_0$  are evaluated at  $\omega t = 0$ . These equations are a linear first-order solution to the exact equations of motion. Equations (7) are readily applied to a two-impulse rendezvous problem since it is possible to define explicitly the required initial velocity for intercept and the resulting velocity at intercept. However, equations (7) are valid only for small values of  $|\bar{y}|$  and small transfer angles  $\omega t$ .

Since equations (7) permit closed-form solutions, it was felt that an effort must be made to improve this type of equation for rendezvous application. To this end, equations (7) were examined in a special way in an effort to define the cause of their inaccuracy. A possible way to improve the equations became evident after examination. Although the improvement lacked mathematical rigor, the approximate equations

(obtained from eqs. (7)) were found to be generally more accurate than equations (7) for rendezvous application.

### Examination of the Approximate-Solution Equations

The rendezvous trajectory may be considered as a trajectory which has been perturbed about a circular orbit at the initial relative altitude  $\bar{y} = \bar{y}_0$ . If the eccentricity of the trajectory is small, the circular part is of primary significance, and it is important to know how this part is mathematically expressed by equations (7). This may be determined by first considering  $\dot{\bar{x}}_0$  to be composed of two parts, with one part  $\dot{\bar{x}}_c$  being the value of  $\dot{\bar{x}}_0$  for a circular orbit at  $\bar{y}_0$ . Thus

$$\dot{\bar{x}}_0 = \dot{\bar{x}}_c + \Delta\dot{\bar{x}}_0 \quad (8)$$

Substituting equation (8) into equations (7) and rearranging the terms leads to the following equations:

$$\left. \begin{aligned} \bar{x} &= \left[ (4 \sin \omega t - 3\omega t)\dot{\bar{x}}_c + 6(\omega t - \sin \omega t)\bar{y}_0 + \bar{x}_0 \right] \\ &\quad + \left[ (4 \sin \omega t - 3\omega t)\Delta\dot{\bar{x}}_0 + 2(1 - \cos \omega t)\dot{\bar{y}}_0 \right] \\ \bar{y} &= \left[ -2(1 - \cos \omega t)\dot{\bar{x}}_c + (4 - 3 \cos \omega t)\bar{y}_0 \right] \\ &\quad + \left[ -2(1 - \cos \omega t)\Delta\dot{\bar{x}}_0 + (\sin \omega t)\dot{\bar{y}}_0 \right] \end{aligned} \right\} \quad (9)$$

The terms are arranged so that if the initial conditions correspond to a circular orbit at the initial altitude, which requires that  $\Delta\dot{\bar{x}}_0 = \dot{\bar{y}}_0 = 0$ , the second bracketed terms of the  $\bar{x}$  and  $\bar{y}$  equations vanish. The first bracketed terms then determine in a linear sense  $\bar{x}$  and  $\bar{y}$  as a function of  $\omega t$  for the ferry being initially in a circular orbit. The terms are functions of  $\bar{x}_0$  and  $\bar{y}_0$  (both of which may presumably be measured),  $\dot{\bar{x}}_c$  (which is computed), and  $\omega t$ .

Exact expressions for  $\bar{x}_c$ ,  $\dot{\bar{x}}_c$ , and  $\bar{y}_c$  under the circular orbit assumptions are

$$\left. \begin{aligned} \bar{x}_c &= \dot{\bar{x}}_c \omega t + \bar{x}_0 \\ \bar{y}_c &= \bar{y}_0 \end{aligned} \right\} \quad (10a)$$

$$\dot{\bar{x}}_c = 1 - \left(1 + \bar{y}_0\right)^{-3/2} \quad (10b)$$



Equation (10b) is determined from equations (4) by setting  $\ddot{\bar{y}}$  equal to 0. When comparing equations (10a) with the first bracketed terms of equations (9) it is difficult to see the error source. A more meaningful comparison between these equations is possible if  $\dot{\bar{x}}_c$  is expressed as a function of  $\bar{y}_0$ . By expanding  $(1 + \bar{y}_0)^{-3/2}$  in a binomial series, equation (10b) may be written as

$$\dot{\bar{x}}_c = \frac{3}{2} \bar{y}_0 - Q(\bar{y}_0) \quad (11)$$

where  $Q(\bar{y}_0)$  is used to represent the second-order and higher order terms of the expansion.

When equation (11) is substituted into equations (10a) and into the first bracketed terms of equations (9), and the latter equations are subtracted from the former, expressions for the error are obtained:

$$\left. \begin{aligned} (\bar{x}_c)_{\text{error}} &= 4Q(\bar{y}_0) (\sin \omega t - \omega t) \\ (\bar{y}_c)_{\text{error}} &= -2Q(\bar{y}_0) (1 - \cos \omega t) \end{aligned} \right\} \quad (12)$$

The error thus depends upon the higher order terms of  $\bar{y}_0$  and of  $\omega t$ . No error is present if the ferry vehicle is initially at the same altitude as the target vehicle. The  $(\bar{x}_c)_{\text{error}}$  term is noncyclic and the error increases with  $\omega t$ . The  $(\bar{y}_c)_{\text{error}}$  term is cyclic and the error is zero for transfer angles that are multiples of  $2\pi$  radians. The maximum value of  $(\bar{y}_c)_{\text{error}}$  occurs at  $(2k - 1)\pi$  radians ( $k$  is an integer). Although it is possible to approximate  $Q(\bar{y}_0)$  by using only its lowest order term  $\frac{15}{2} \bar{y}_0^2$ , the results presented in this paper are based upon the exact value of  $Q(\bar{y}_0)$ . The exact expression is

$$Q(\bar{y}_0) = \frac{3}{2} \bar{y}_0 - 1 + (1 + \bar{y}_0)^{-3/2} \quad (13)$$

which is determined by equating equation (10b) to equation (11).

Some example calculations will better illustrate equations (12). A lunar orbit is assumed, with the reference or target orbit located at an altitude of 148.16 kilometers (80 nautical miles) above the lunar surface. The ferry vehicle is initially located at different altitudes from the target vehicle and its initial velocity is that required for a circular orbit. The errors computed from equations (12) for these conditions are illustrated in figure 2, for six initial altitudes of the ferry vehicle. Although the case

$y_0 = -250$  kilometers is physically impossible, it is included for completeness. Figure 2(a) shows that  $(x_c)_{\text{error}}$  increases sharply with both  $y_0$  and  $\omega t$ , and that the errors are relatively large for the longer transfer angles. Although the  $(y_c)_{\text{error}}$  term, figure 2(b), is smaller than the corresponding  $(x_c)_{\text{error}}$  term, it is still significant. On the basis of equations (12) and figure 2, the predicted values of  $x$  and  $y$  for any value of  $\omega t$  tend to be larger than they should be. For example, the ferry vehicle initially located in a circular orbit at  $y_0 = -100$  kilometers will rise to an altitude of  $-60$  kilometers at  $\omega t = 180^\circ$ . The value of  $x$  is about 135 kilometers greater than it should be for this case.

Figure 2 has shown that large errors occur in the first bracketed terms of equations (9) for large transfer angles when a circular orbit is assumed. Equations (9) when applied to rendezvous are also inaccurate for large transfer angles. These two facts suggest that the inaccuracy of equations (9) may be due to the inaccuracy of their first bracketed terms, particularly when  $\bar{y}_0$  is large. This line of reasoning also suggests that better results may be obtained for large transfer angles if the inaccurate first bracketed terms are replaced by the exact terms of equations (10a). This approach will be used in deriving a new set of equations of relative motion.

#### Derivation of Modified Equations of Motion

The first bracketed terms of equations (9) are replaced by their exact expressions from equations (10a), and  $\Delta\dot{\bar{x}}_0$  is eliminated from the second bracketed terms of equations (9) through the use of equation (8). The resulting equations depend upon the initial conditions  $\bar{x}_0$ ,  $\bar{y}_0$ ,  $\dot{\bar{x}}_0$ , and  $\dot{\bar{y}}_0$ , and upon  $\dot{\bar{x}}_c$ . The  $\dot{\bar{x}}_c$  terms may now be replaced by equation (11) so that the resulting equations will depend upon only the initial-condition terms and  $\omega t$ . When these operations are performed and the equations are simplified, the new modified equations of motion are obtained:

$$\left. \begin{aligned} \bar{x} &= (4 \sin \omega t - 3\omega t)\dot{\bar{x}}_0 + 2(1 - \cos \omega t)\dot{\bar{y}}_0 + 6(\omega t - \sin \omega t)\bar{y}_0 + \bar{x}_0 + 4Q(\bar{y}_0)(\sin \omega t - \omega t) \\ \bar{y} &= -2(1 - \cos \omega t)\dot{\bar{x}}_0 + (\sin \omega t)\dot{\bar{y}}_0 + (4 - 3 \cos \omega t)\bar{y}_0 - 2Q(\bar{y}_0)(1 - \cos \omega t) \end{aligned} \right\} \quad (14)$$

where  $Q(\bar{y}_0)$  is defined by equation (13). Equations (14) are the solution to the differential equations

$$\left. \begin{aligned} \ddot{\bar{x}} &= 2\dot{\bar{y}} \\ \ddot{\bar{y}} &= -2\dot{\bar{x}} + 3\bar{y} - 2Q(\bar{y}_0) \end{aligned} \right\} \quad (15)$$

The difference between equations (15) and equations (6) may be noted.

Equations (14) differ from the original equations (7) by including the terms involving  $Q(\bar{y}_0)$ , which depend only on  $\bar{y}_0$  and  $\omega t$ .

#### Derivation of Two-Impulse Equations for Rendezvous

In order to determine the first and second impulses, it is necessary to know the initial required velocity components for intercept and the final velocity components at intercept. The initial required velocity components  $\dot{\bar{x}}_0^*$  and  $\dot{\bar{y}}_0^*$  are determined by setting  $\bar{x} = \bar{y} = 0$  in equations (14) (intercept condition) and solving for the initial conditions  $\dot{\bar{x}}_0$  and  $\dot{\bar{y}}_0$ . The following equations are obtained:

$$\left. \begin{aligned} \dot{\bar{x}}_0^* &= \frac{1}{\delta} \left[ \bar{x}_0 \sin \omega \tau + \bar{y}_0 (6\omega \tau \sin \omega \tau - 14 + 14 \cos \omega \tau) \right] - \frac{Q(\bar{y}_0)}{\delta} (\omega \tau \sin \omega \tau + \delta) \\ \dot{\bar{y}}_0^* &= \frac{1}{\delta} \left[ 2(1 - \cos \omega \tau) \bar{x}_0 + \bar{y}_0 (4 \sin \omega \tau - 3\omega \tau \cos \omega \tau) \right] + \frac{Q(\bar{y}_0)}{4\delta} (\delta - 3\omega \tau \sin \omega \tau) \omega \tau \end{aligned} \right\} \quad (16)$$

where

$$\delta = 3\omega \tau \sin \omega \tau - 8(1 - \cos \omega \tau) \quad (17)$$

Equations (16) determine the initial required velocity components for intercept as a function of the initial displacement  $\bar{x}_0, \bar{y}_0$  and the desired transfer angle to intercept  $\omega t = \omega \tau$ . The velocity components at intercept are determined by using equations (16) as the initial velocity components of the trajectory. When equations (16) are substituted into the velocity equations defined by differentiating equations (14) with respect to  $\omega t$ , the velocity components at intercept may be obtained as

$$\left. \begin{aligned} \dot{\bar{x}}(\omega \tau) &= \frac{1}{\delta} \left[ \bar{x}_0 \sin \omega \tau + 2\bar{y}_0 (1 - \cos \omega \tau) \right] - \frac{Q(\bar{y}_0)}{\delta} (\omega \tau \sin \omega \tau + \delta) \\ \dot{\bar{y}}(\omega \tau) &= \frac{1}{\delta} \left[ -2\bar{x}_0 (1 - \cos \omega \tau) + \bar{y}_0 (4 \sin \omega \tau - 3\omega \tau) \right] - \frac{Q(\bar{y}_0)}{4\delta} (\delta - 3\omega \tau \sin \omega \tau) \omega \tau \end{aligned} \right\} \quad (18)$$

The first impulse is the vector difference between the initial required velocity for intercept and the actual initial velocity. The magnitude of the first impulse is defined by the dimensional equation

$$\Delta V_1 = \sqrt{\left[ \Delta(r_{F,o} \dot{\sigma}_{F,o}) \right]^2 + \left[ \Delta(\dot{r}_{F,o}) \right]^2} \quad (19)$$

where  $\Delta(r_{F,o} \dot{\sigma}_{F,o})$  is the change in the circumferential velocity of the ferry vehicle and  $\Delta(\dot{r}_{F,o})$  is the change in its radial velocity, or

$$\left. \begin{aligned} \Delta(r_{F,o} \dot{\sigma}_{F,o}) &= r_{F,o} (\dot{\sigma}_{F,o}^* - \dot{\sigma}_{F,o}) \\ \Delta(\dot{r}_{F,o}) &= \dot{r}_{F,o}^* - \dot{r}_{F,o} \end{aligned} \right\} \quad (20)$$

Substitutions from equations (1) lead to

$$\left. \begin{aligned} \Delta(r_{F,o} \dot{\sigma}_{F,o}) &= r_{F,o} \left[ \left( \omega - \frac{\dot{x}_o^*}{r_T} \right) - \left( \omega - \frac{\dot{x}_o}{r_T} \right) \right] = -\frac{r_{F,o}}{r_T} (\dot{x}_o^* - \dot{x}_o) = -\left( 1 + \frac{y_o}{r_T} \right) (\dot{x}_o^* - \dot{x}_o) \\ \Delta(\dot{r}_{F,o}) &= \dot{y}_o^* - \dot{y}_o \end{aligned} \right\} \quad (21)$$

The first-impulse equation becomes, in nondimensional form,

$$\Delta \bar{V}_1 = \sqrt{\left[ \left( 1 + \bar{y}_o \right) (\dot{\bar{x}}_o^* - \dot{\bar{x}}_o) \right]^2 + \left[ \dot{\bar{y}}_o^* - \dot{\bar{y}}_o \right]^2} \quad (22)$$

Equation (22) represents the actual magnitude of the first impulse applied, expressed in shell coordinates. Since the relative velocity between the ferry and target vehicles must be brought to zero for rendezvous, the second impulse is simply the vector difference between the zero velocity desired and the intercept velocity. The magnitude of the second impulse in nondimensional form is

$$\Delta \bar{V}_2 = \sqrt{\left[ \dot{\bar{x}}(\omega\tau) \right]^2 + \left[ \dot{\bar{y}}(\omega\tau) \right]^2} \quad (23)$$

Since the second impulse is executed at  $\bar{y} = 0$ , a term involving  $\bar{y}_o$  does not appear in the second-impulse equation as it did in the first. This statement may be verified by the procedure used in determining equation (22).

## RESULTS AND DISCUSSION

In order to demonstrate the effectiveness of using the new approximate equations, numerous intercept trajectories were computed. The ferry vehicle was assumed to be initially located at some slant-range distance  $d_o$  from the target vehicle. For each

given  $d_0$  value, the ferry vehicle's initial direction with respect to the target vehicle (defined by angle  $\phi_0$ , fig. 1) was varied from  $0^\circ$  to  $360^\circ$  in increments of  $30^\circ$ . From each initial relative polar coordinate position  $(d_0, \phi_0)$  of the ferry vehicle the initial required velocity for intercept was computed by two methods for transfer angles from  $30^\circ$  to  $330^\circ$ . These computed values of required velocity were then used as the input to an exact-equation computational program which determined  $d_R$ , the slant-range separation of the two vehicles at the intended intercept. Method B uses equations (16). Method A uses the Clohessy and Wiltshire equations, equations (16) with  $Q(\bar{y}_0) = 0$ . The resulting  $d_R$  values for both methods were compared. Rendezvous maneuvers were assumed to occur about only one central body, the moon. The target vehicle's circular orbit was located at an altitude of 148.16 kilometers (80 nautical miles).

The results of the investigation (for the two methods) are presented in figures 3 and 4 as semilog plots. The separation at intercept  $d_R$  is plotted against the transfer angle  $\omega\tau$  for a range of initial polar angles  $\phi_0$ . For figure 3, a  $d_0$  value of 100 kilometers was used for  $\phi_0$  values of  $0^\circ$  to  $330^\circ$ . Figure 4 shows the result for  $d_0$  values of 50 kilometers and 250 kilometers for a single  $\phi_0$  of  $90^\circ$ . In figures 3 and 4 faired curves were used to connect the data points.

For  $\bar{y}_0 \approx 0$  (figs. 3(a) and 3(g), where  $\phi_0 = 0^\circ$  and  $180^\circ$ ), both methods result in about the same values of  $d_R$  for any given  $\omega\tau$ . The magnitude of  $d_R$  is small compared with the initial  $d_0$  value. As  $|\bar{y}_0|$  increases toward its maximum ( $\phi_0 = 90^\circ$  and  $270^\circ$ ), the  $d_R$  obtained by the use of method A increases sharply with increasing  $\omega\tau$ . Method B, however, does not exhibit this large increase in  $d_R$ . For all of figure 3, with the exception of  $\phi_0 = 0^\circ$  and  $180^\circ$ , method B results in smaller  $d_R$  values than method A over the  $\omega\tau$  range of  $120^\circ$  to  $300^\circ$ . In most cases shown, this is true for even smaller  $\omega\tau$  values than  $120^\circ$ . In the region where method A results in smaller  $d_R$  values than method B (small values of  $\omega\tau$ ) the differences in  $d_R$  are small. However, in the region where method B is superior to method A ( $\omega\tau$  not small) the differences in  $d_R$  values are usually large. The values in the  $\phi_0$  interval of  $0^\circ$  to  $180^\circ$  are approximately the same as the values in the interval from  $180^\circ$  to  $360^\circ$ . That is, the curves of the  $\phi_0 = 30^\circ$  case are similar to those of the  $\phi_0 = 210^\circ$  case.

The two methods were also compared for  $d_0$  values from 25 to 750 kilometers, but the results are not presented here. The comparison between methods A and B remained essentially the same as that presented in figure 3. As might be expected, the larger the  $d_0$  value used, the larger the resulting  $d_R$  values. This is illustrated in figure 4 for  $d_0$  values of 50 and 250 kilometers with  $\phi_0 = 90^\circ$ . The curves of figure 4(a) are similar to those of figure 4(b), and also similar to those of figure 3(d), where  $d_0 = 100$  kilometers and  $\phi_0 = 90^\circ$ . Both methods deteriorate with increasing  $d_0$ . When  $d_0$  is increased by a factor of 5, from 50 to 250 kilometers, the maximum

$d_R$  of method A increases from 60 to 1200 kilometers, a factor of 20. The near steady-state interval of method B increases also by a factor of about 20 from about 3 kilometers to about 70 kilometers.

### CONCLUDING REMARKS

The desirability of obtaining simplified and accurate relative equations of motion for two-impulse rendezvous application resulted in the examination of the Clohessy and Wiltshire type of simplified equation. An analysis of these equations revealed that large errors were possible if the radial separation of the two vehicles was large. In an effort to develop more accurate equations for rendezvous, the apparent erroneous terms were replaced by exact terms. The resulting modified equations contained terms not present in the Clohessy and Wiltshire equations. When the modified equations were applied to the problem of rendezvous, they were found to be generally superior to the Clohessy and Wiltshire equations, particularly for long transfer angles.

Langley Research Center,  
National Aeronautics and Space Administration,  
Langley Station, Hampton, Va., March 11, 1968,  
127-51-01-02-23.

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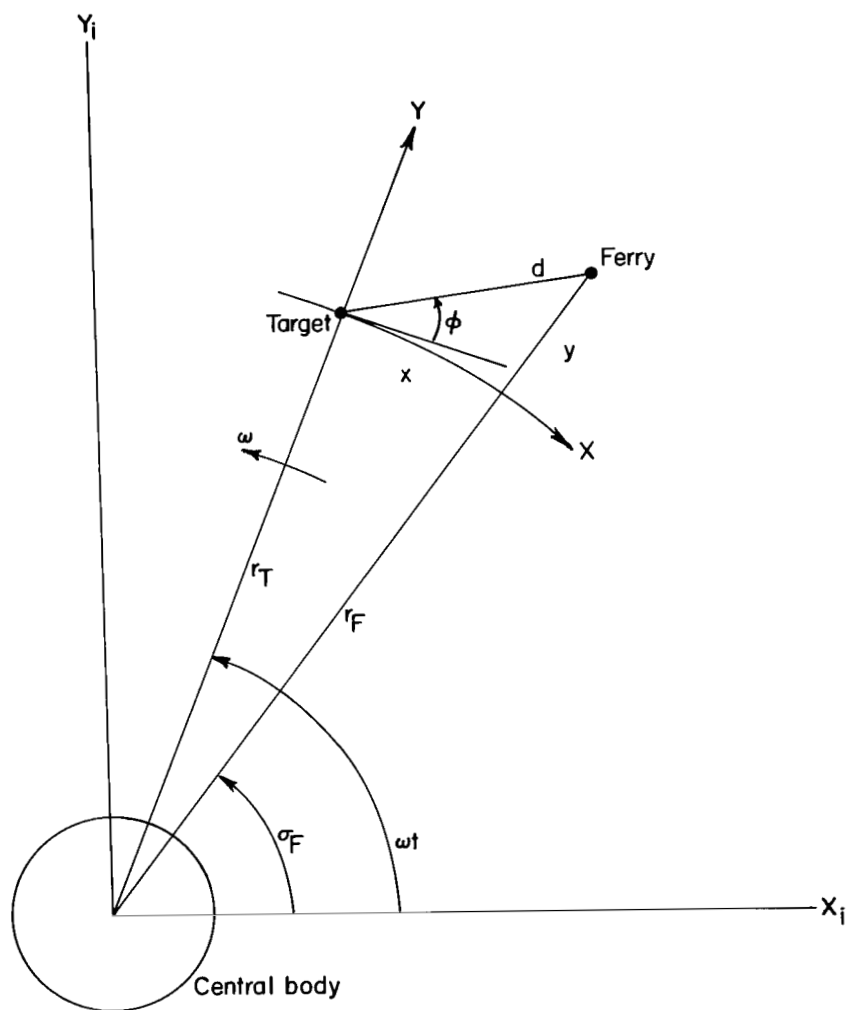
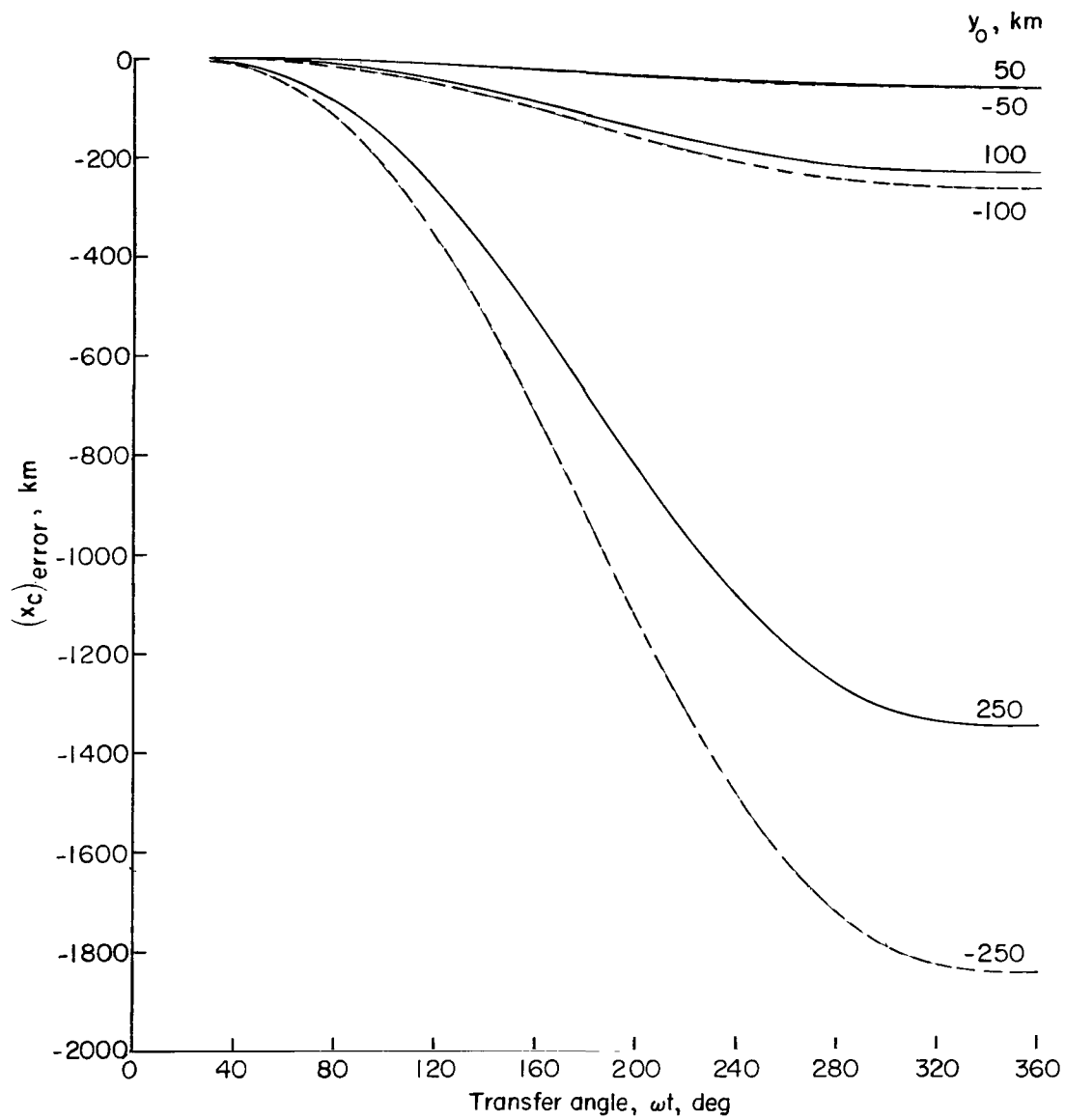


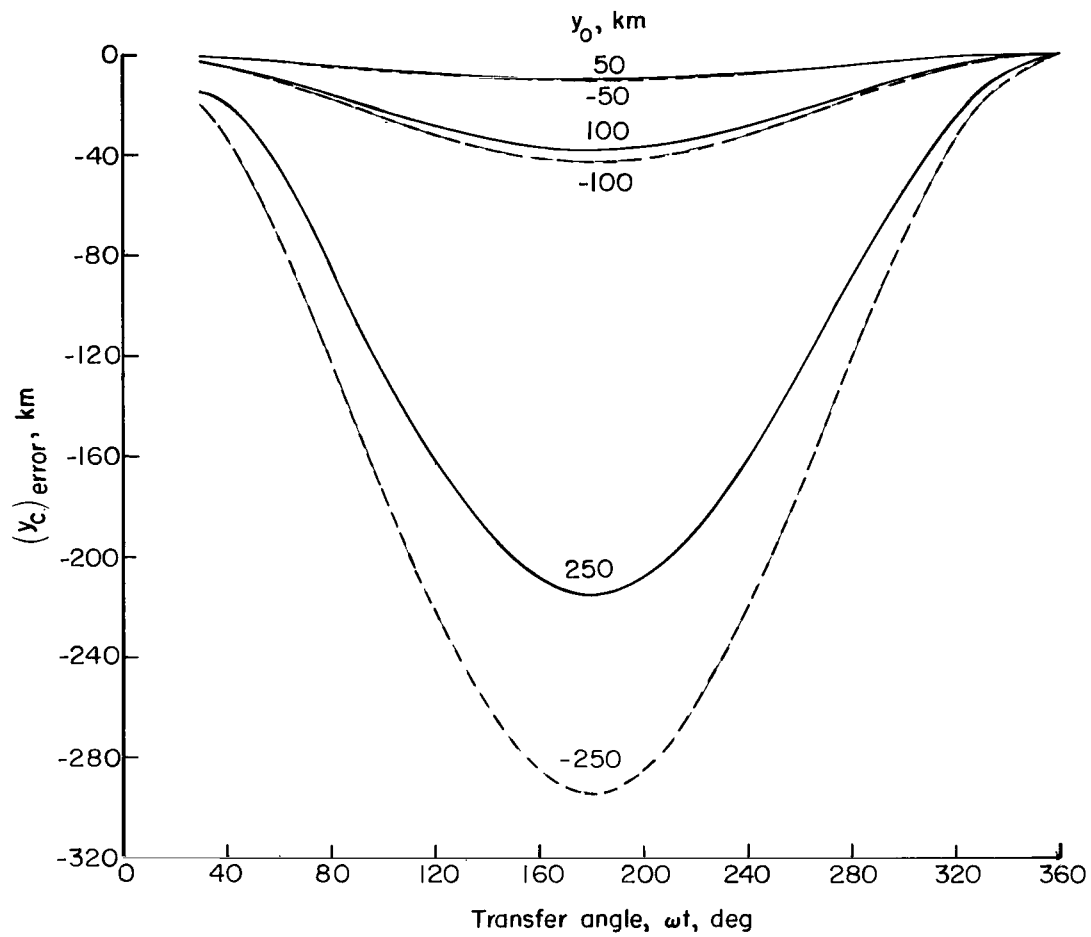
Figure 1.- Coordinate system.



(a) Dimensionalized error in the first bracketed term of the  $\bar{x}$  equation (eq. (9)).

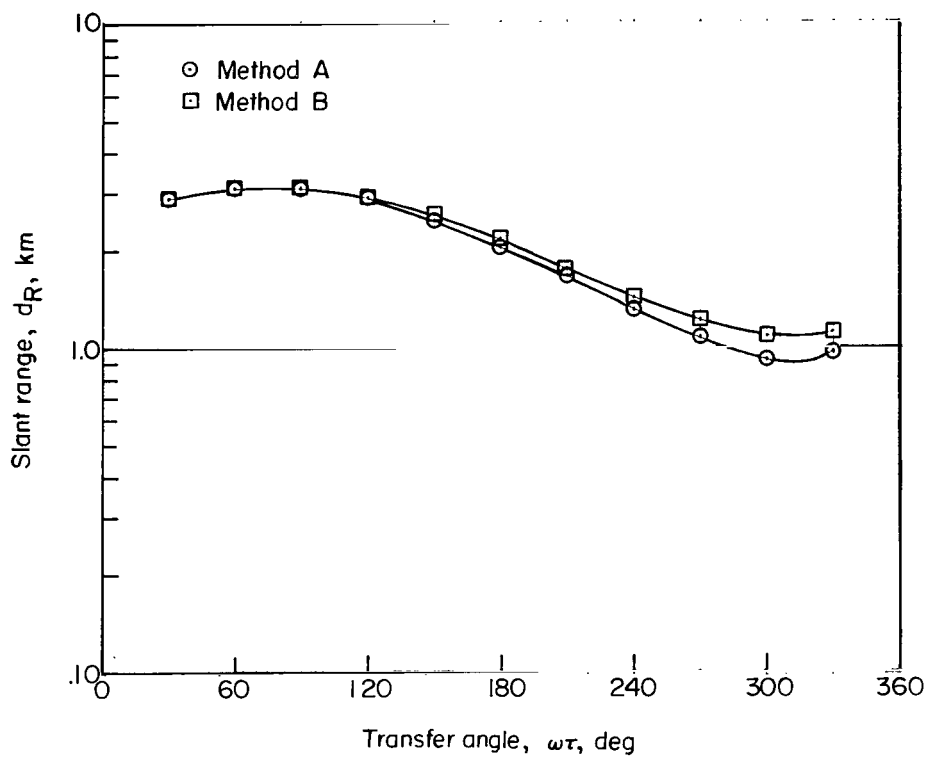
Figure 2.- Position errors of the Clohessy-Wiltshire solution for special initial conditions (the ferry vehicle is in a circular orbit at  $y = y_0$ ).





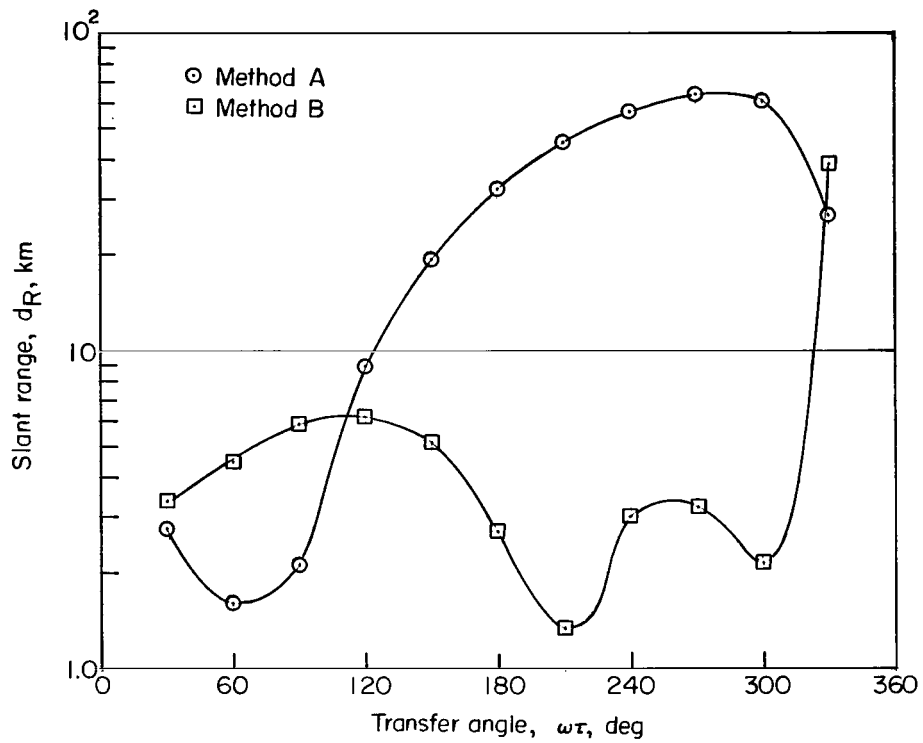
(b) Dimensionalized error in the first bracketed term of the  $\bar{y}$  equation (eq. (9)).

Figure 2.- Concluded.



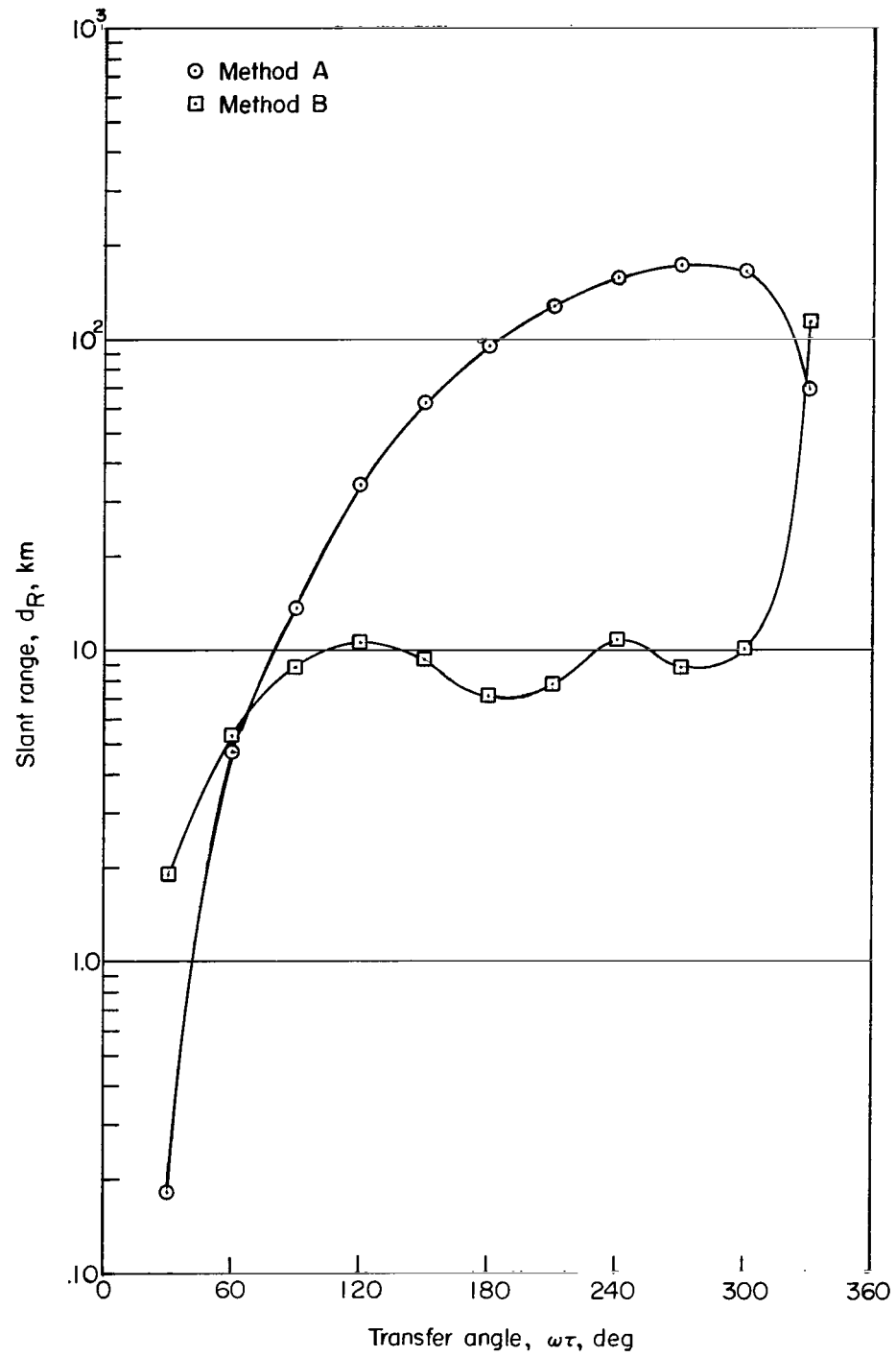
(a)  $\Phi_0 = 0^\circ$ .

Figure 3.- Separation of target and ferry vehicles at intercept resulting from the use of Clohessy-Wiltshire equations (method A) and modified equations (method B), for transfer angles from  $30^\circ$  to  $330^\circ$  and initial orientation angles  $\Phi_0$  from  $0^\circ$  to  $330^\circ$ .  $d_0 = 100$  km.



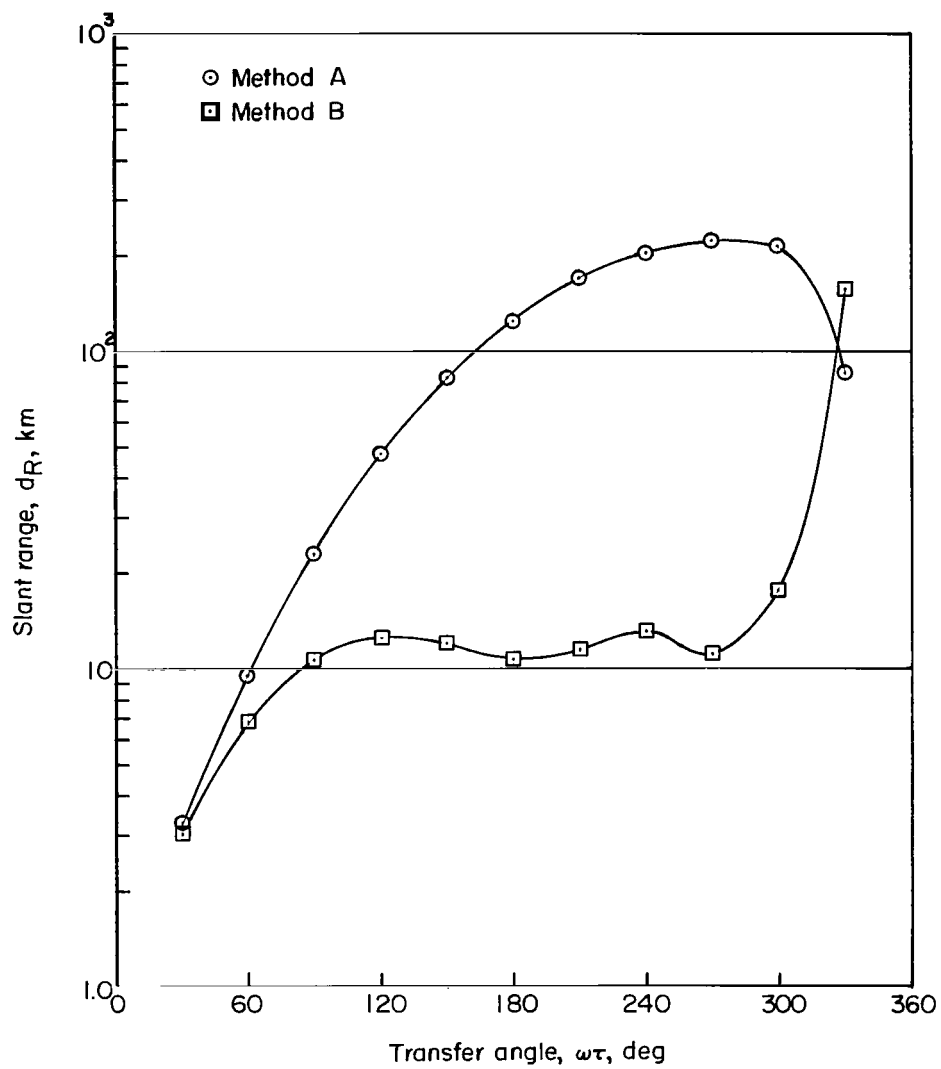
(b)  $\phi_0 = 30^\circ$ .

Figure 3.- Continued.



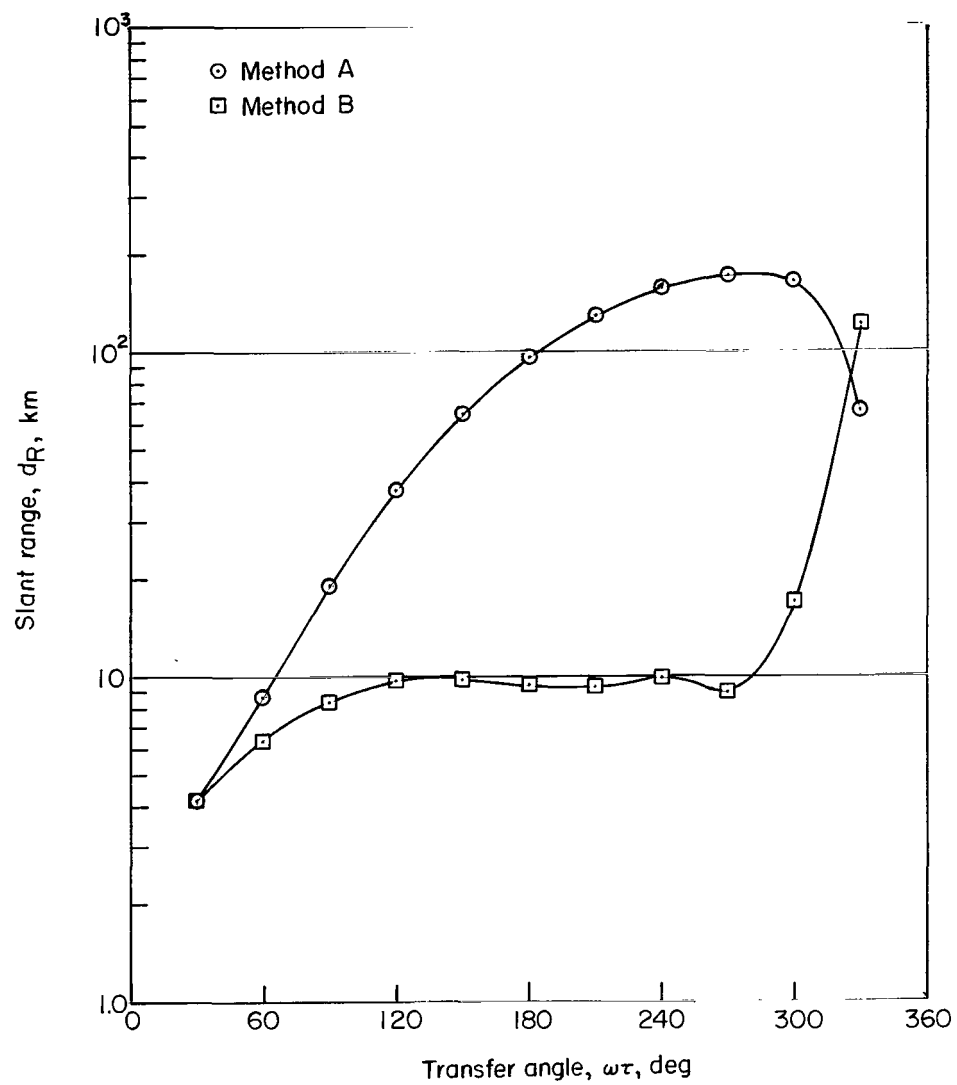
(c)  $\Phi_0 = 60^\circ$ .

Figure 3.- Continued.



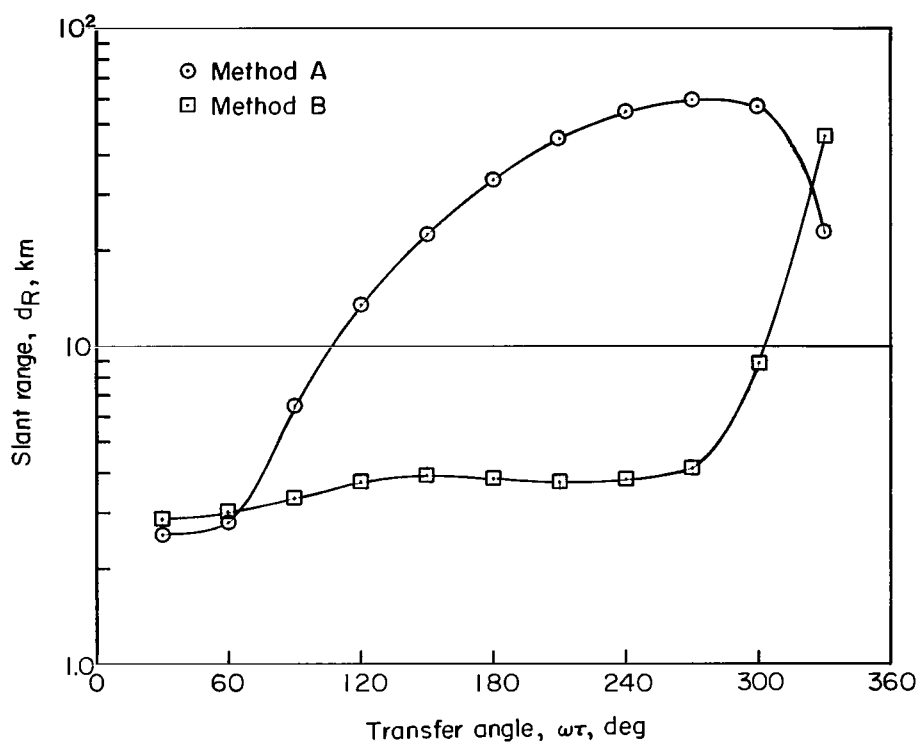
(d)  $\Phi_0 = 90^\circ$ .

Figure 3.- Continued.



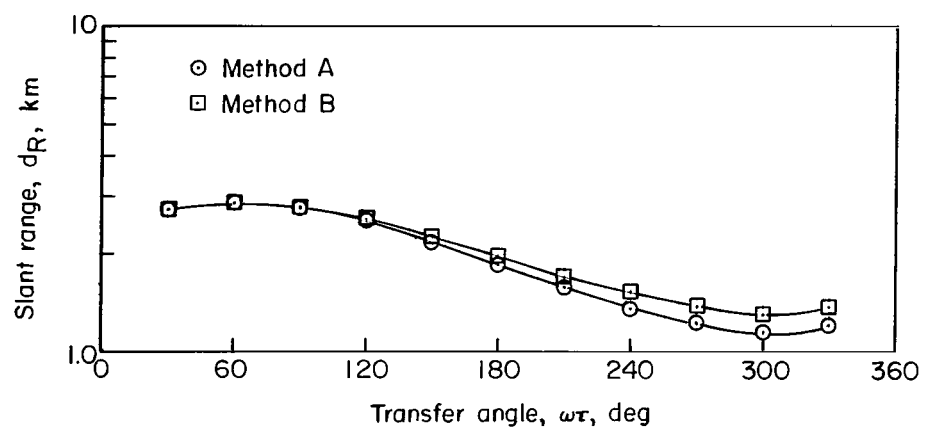
(e)  $\phi_0 = 120^\circ$ .

Figure 3.- Continued.



(f)  $\Phi_0 = 150^\circ$ .

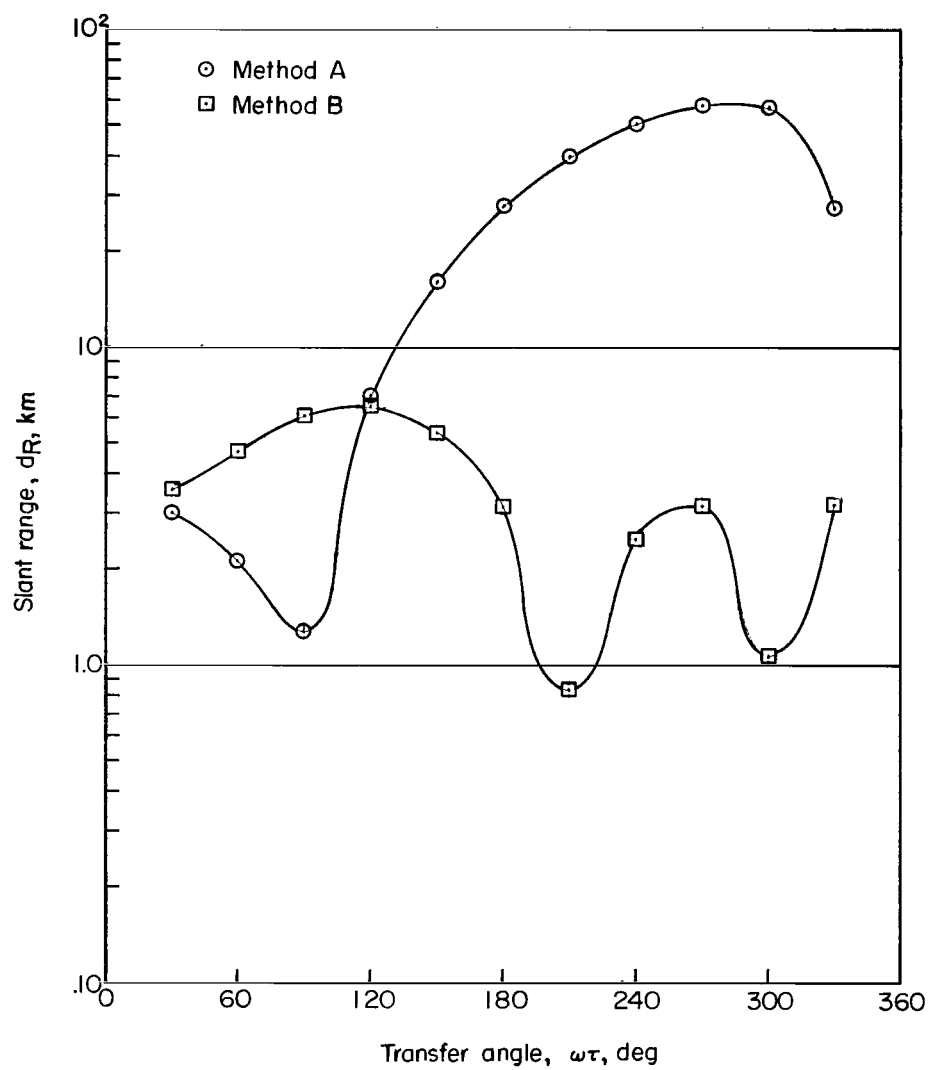
Figure 3.- Continued.



(g)  $\phi_0 = 180^\circ$ .

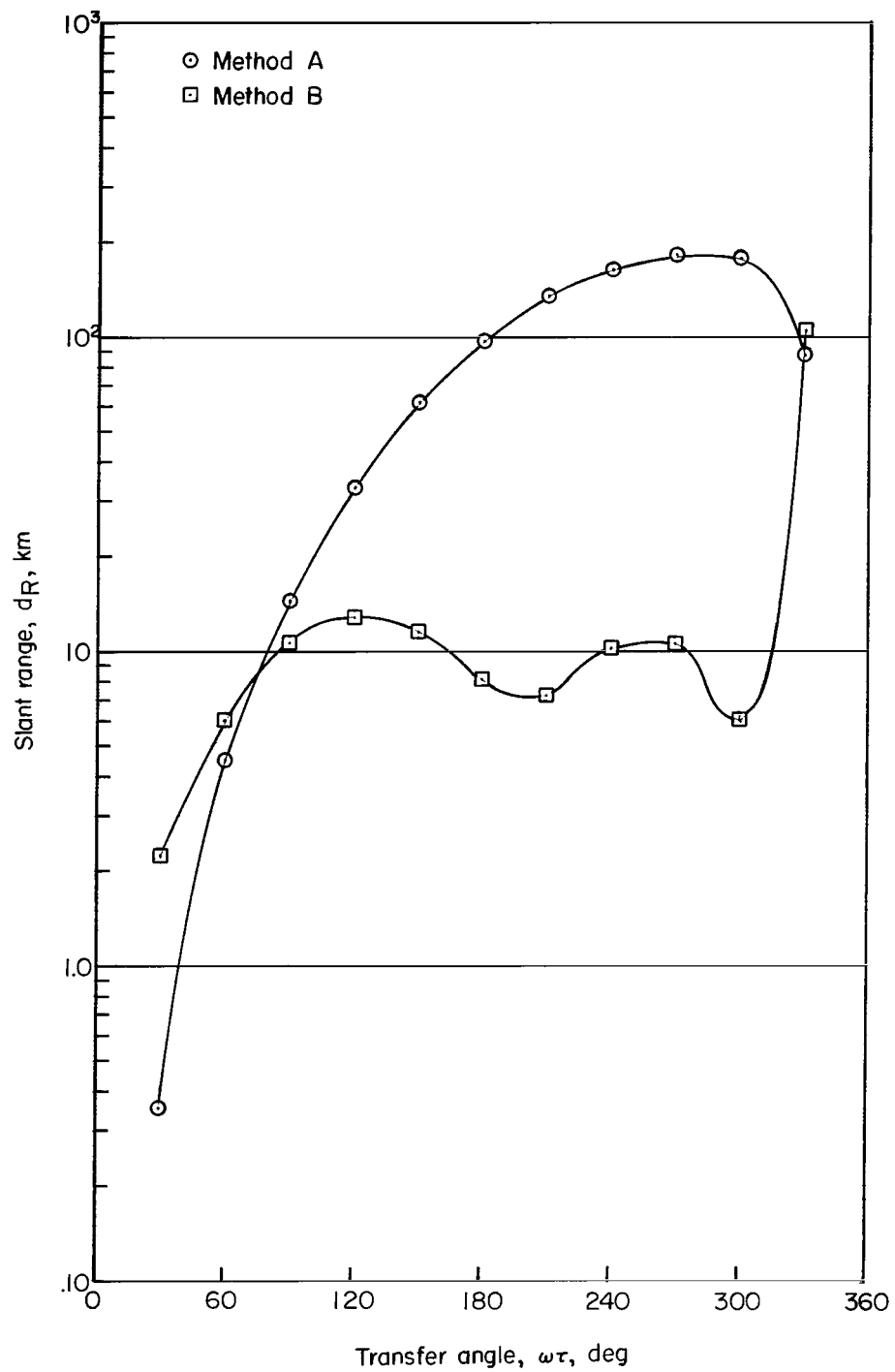
Figure 3.- Continued.





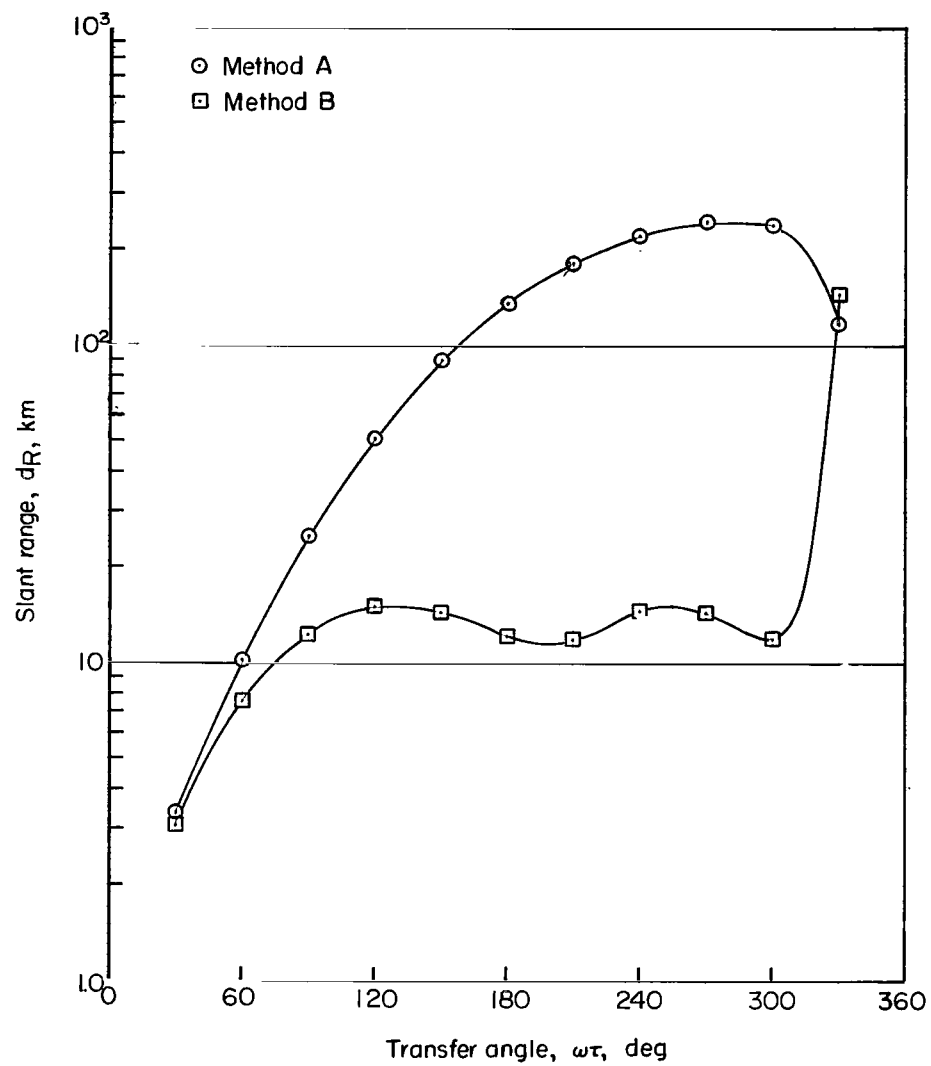
(h)  $\phi_0 = 210^\circ$ .

Figure 3.- Continued.



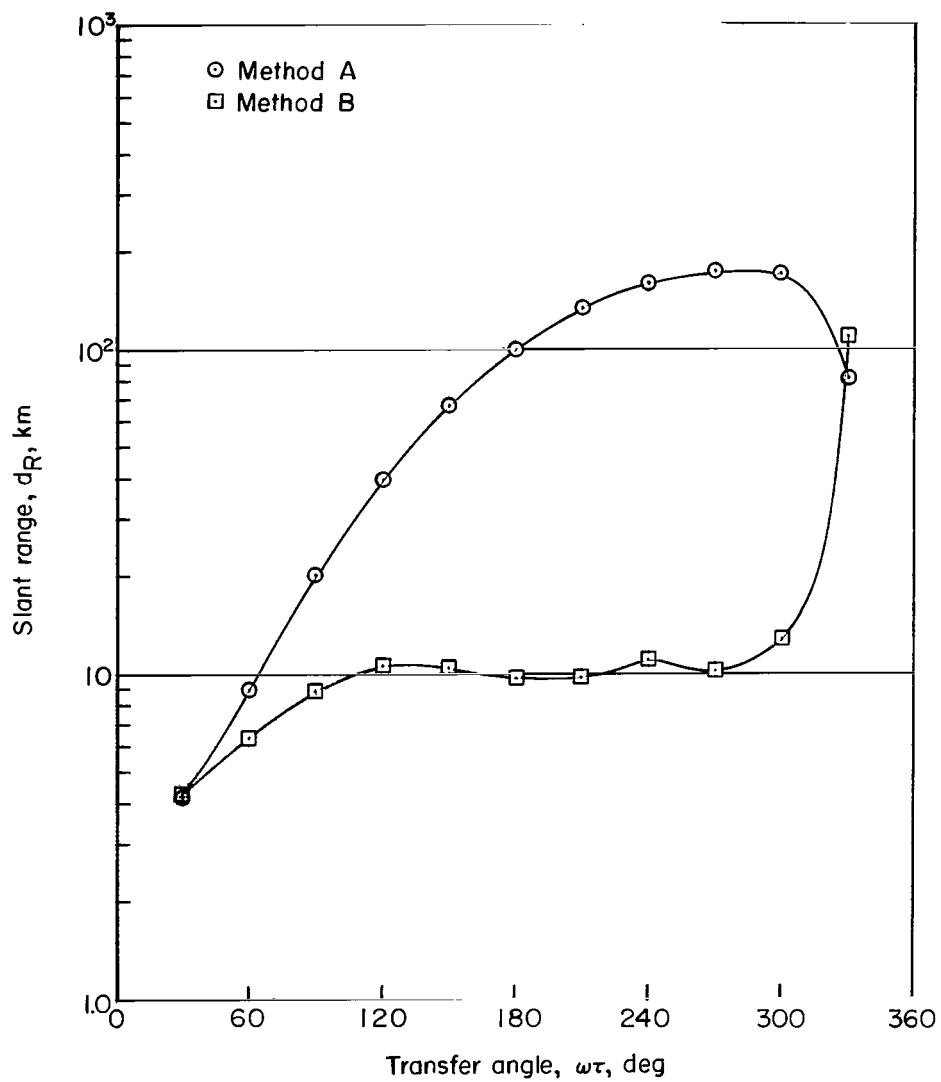
(i)  $\Phi_0 = 240^\circ$ .

Figure 3.- Continued.



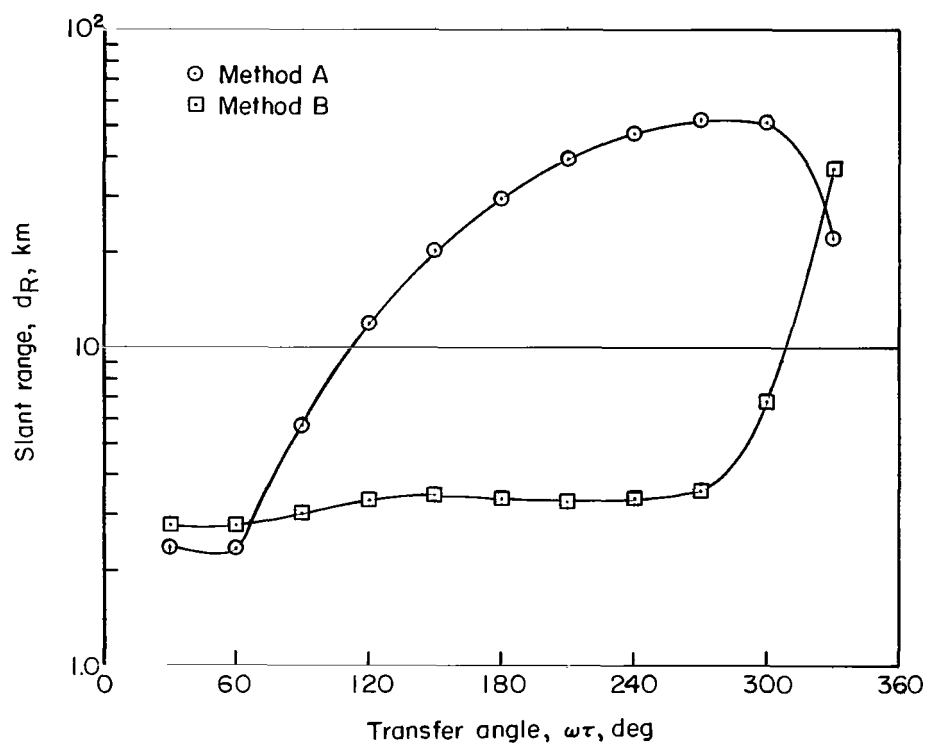
(j)  $\Phi_0 = 270^\circ$ .

Figure 3.- Continued.



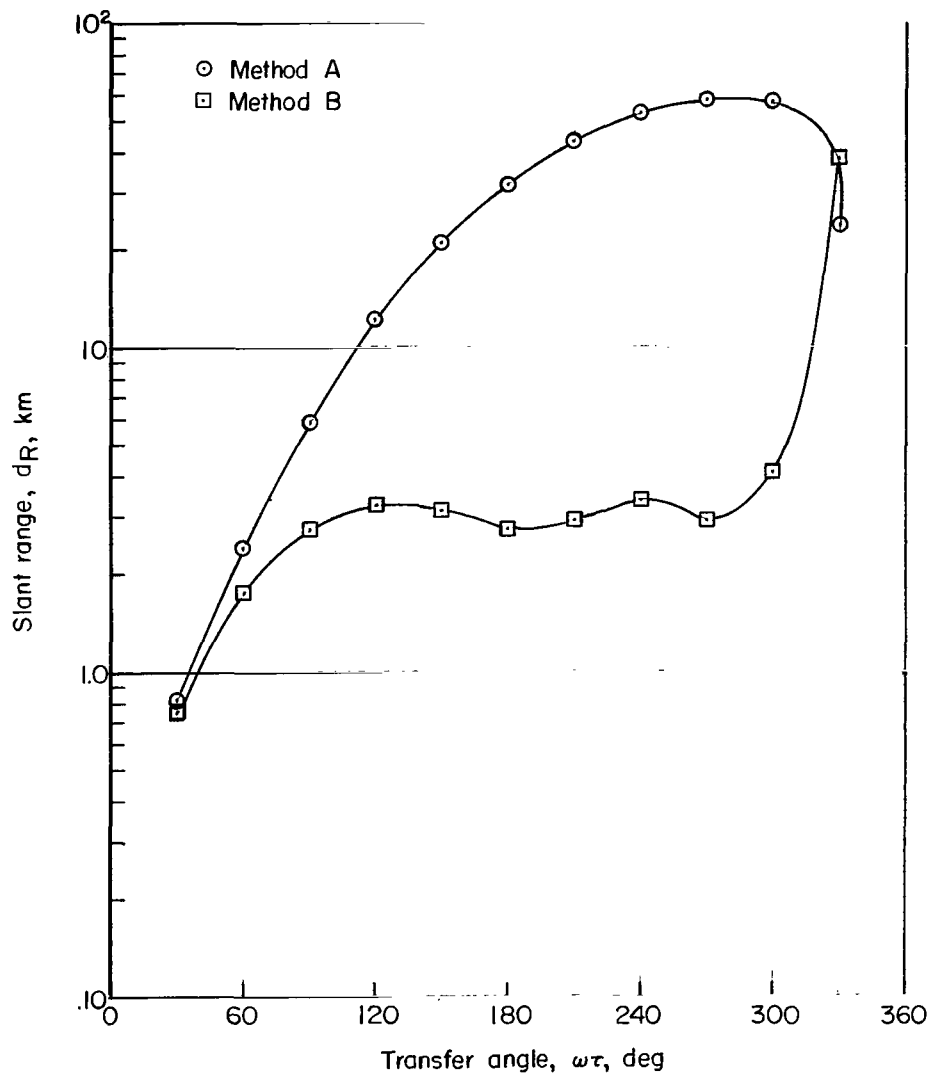
(k)  $\phi_0 = 300^\circ$ .

Figure 3.- Continued.



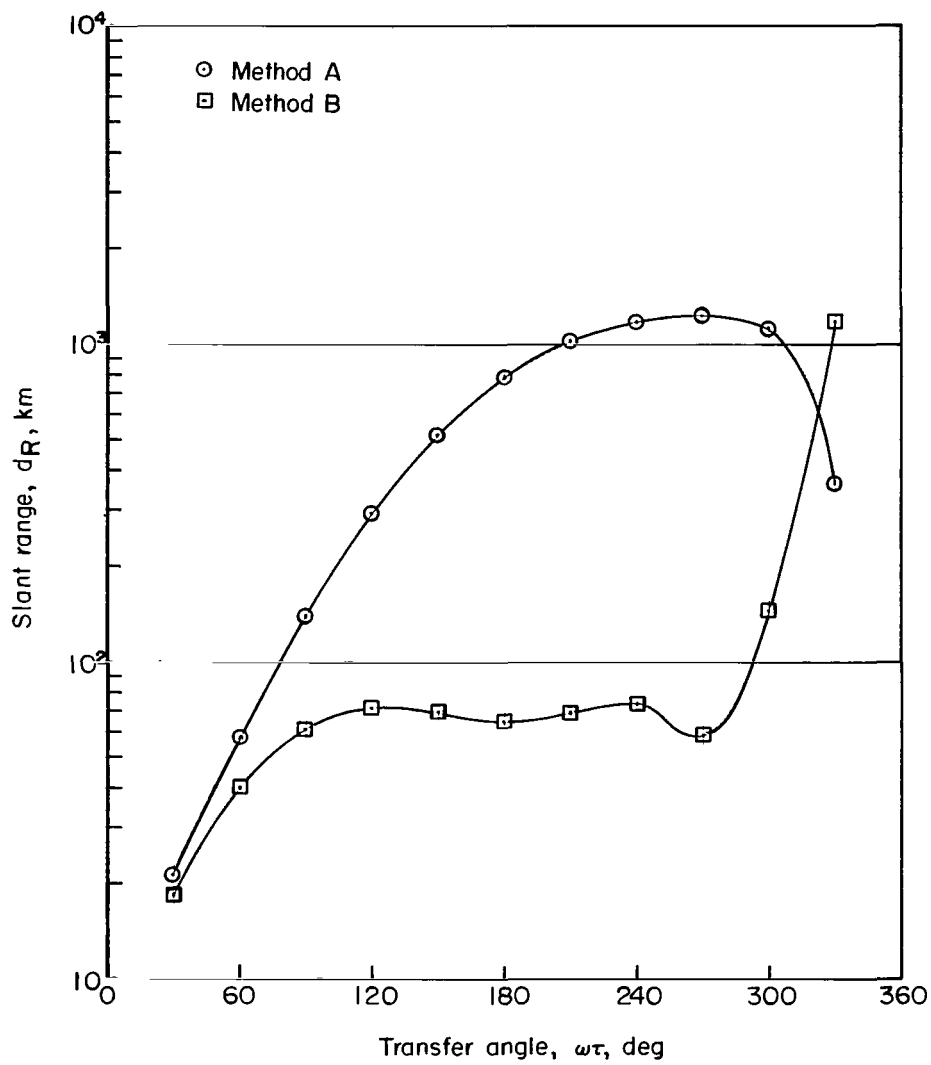
(L)  $\phi_0 = 330^\circ$ .

Figure 3.- Concluded.



(a)  $d_0 = 50$  km.

Figure 4.- Separation of target and ferry vehicles at intercept resulting from the use of Clohessy-Wiltshire equations (method A) and modified equations (method B), for transfer angles from  $30^\circ$  to  $330^\circ$  and initial slant-range separation  $d_0$  of 50 and 250 km.  $\Phi_0 = 90^\circ$ .



(b)  $d_0 = 250$  km.

Figure 4.- Concluded.

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